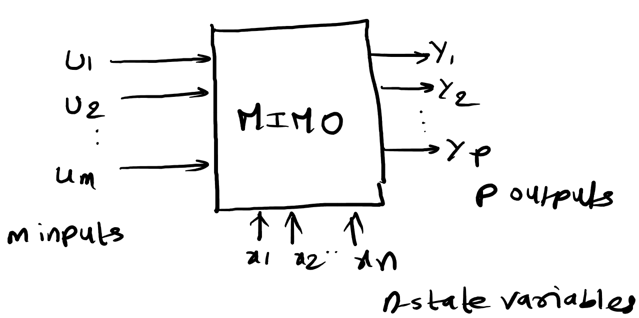
State Space Analysis

Transfer function	State Space Analysis
① It is used for Linear Time Invariant systems.	① It is used for both linear, non linear, both time variant, time invariant.
② It is used for single Input single output system (SISO)	② It is used for both single input single output as well as Multiple inputs and Multiple output systems.
③ Intial conditions are neglected	③ Intial conditions are considered for analysis.
④ Frequency Domain approach	④ Time domain approach.
⑤ Internal variables are not give as feedback	⑤ Internal variables can be give as feedback.
⑥ Transfer function is unique for a system.	⑥ state model of system is not unique.

State space Model Representation:-

consider a Multiple input and Multiple output system



$$\dot{X} = \overset{n\times n}{A}\overset{n\times 1}{X} + \overset{n\times m}{B}\overset{m\times 1}{U} \rightarrow \text{state equation}$$

$$Y = \underset{p\times n}{C}X + \underset{p\times m}{D}U \rightarrow \text{output equation}$$

$U \rightarrow$ Input vector

$Y \rightarrow$ output vector

$X \rightarrow$ state vector

$A \rightarrow$ system Matrix

$B \rightarrow$ Input Matrix

$C \rightarrow$ output Matrix

$D \rightarrow$ state transition Matrix.

$$U \rightarrow \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

$$X \rightarrow \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix}$$

$$\dot{x}_1 = \frac{d}{dt} x_1$$

Problem 1:- Obtain state space model for the system described by differential equation

$$\dddot{y} + 6\ddot{y} + 11\dot{y} + 6y + U = 0$$

Sol:- Identify order of system.

order = 3.

i.e 3 state variables are present

Let x_1, x_2, x_3 be state variables.

$$\text{Let } x_1 = y \quad — ①$$

$$x_2 = \frac{dy}{dt} = \dot{y} \quad — ② \quad \rightarrow \quad x_2 = \dot{x}_1$$

$$x_3 = \frac{d^2y}{dt^2} = \ddot{y} \quad — ③ \quad \rightarrow \quad x_3 = \dot{x}_2$$

$$\dddot{y} = \dot{x}_3 \quad — ④$$

Now substitute ①, ②, ③ & ④ in given system

$$\dot{x}_3 + 6x_3 + 11x_2 + 6x_1 + U = 0$$

$$\dot{X} = AX + BU$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix}$$

$$\dot{x}_3 = -6x_3 - 11x_2 - 6x_1 - U$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

State Equation

$Y = CX + DU$ (o/p Equation)

$$Y = x_1$$

Output Equation

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \end{pmatrix}_{2\times 1}$$

State Equation

$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}_{3\times 1}$

① state Equation $\dot{X} = AX + BU$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}}_{B} U$$

$\leftarrow y = x_1$

② output Equation $y = CX + DU$

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{0}_{D \text{ Matrix}}$$

problem 2 :- $\dddot{y} + 3\ddot{y} + 17\dot{y} + 5y = 10U$

Techniques to convert Transfer function to state space Models

① Direct Decomposition method
② Fosters form / cananical form
③ cascade programming method.
④ signal flow Graph Method.

① Direct Decomposition Method

① In this method the denominator of transfer function is arranged in a specific form and then draw the decomposition diagram for denominator.

② Then obtain state model from the diagram by adding numerator blocks.

Let $T.F = \dfrac{1}{s+a}$

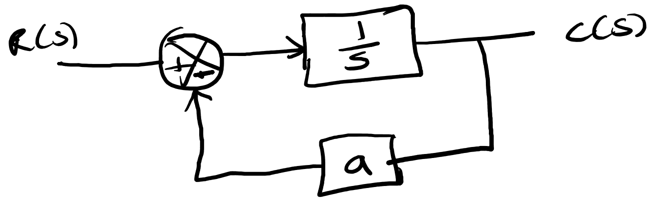
① $\dfrac{1}{s(1+\frac{a}{s})} = \dfrac{\frac{1}{s}}{1+\frac{1}{s}\cdot a}$

Let $G(s) = \frac{1}{s}$
$H(s) = a.$

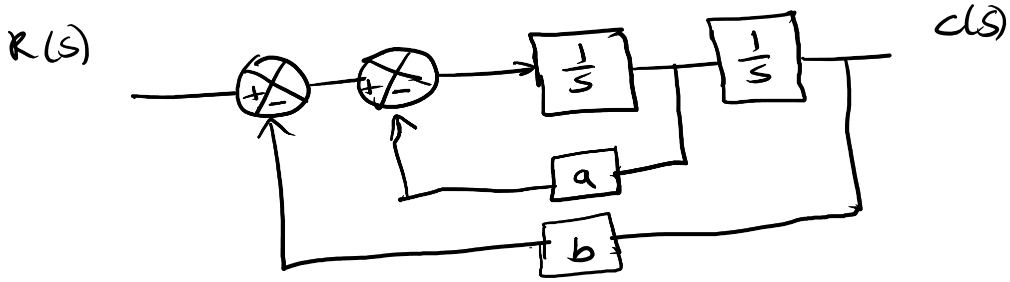
① $\frac{1}{s(1+\frac{a}{s})} = \frac{-}{1+\frac{1}{s}\cdot a}$

$H(s) = a.$

$\rightarrow \frac{G(s)}{1+G(s)H(s)}$



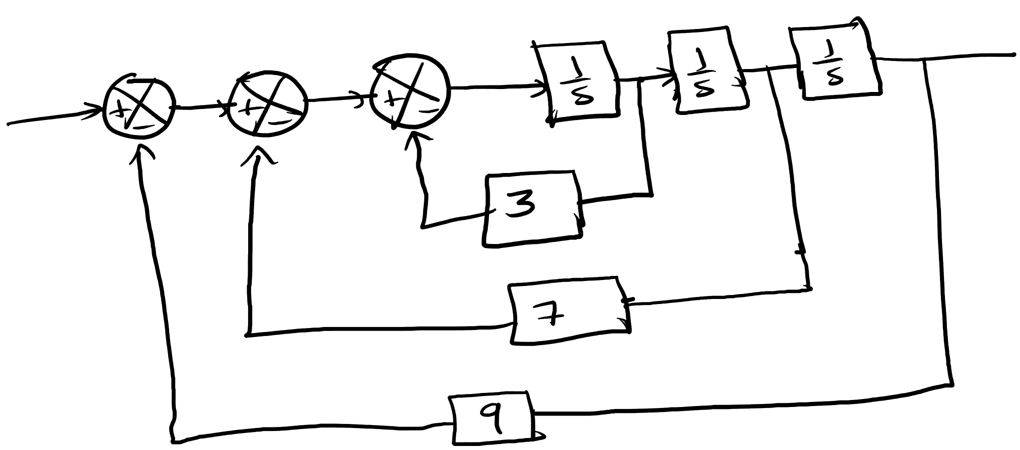
② $\frac{1}{s^2+as+b} = \frac{\frac{1}{s^2}}{1+\frac{a}{s}+\frac{b}{s^2}}$



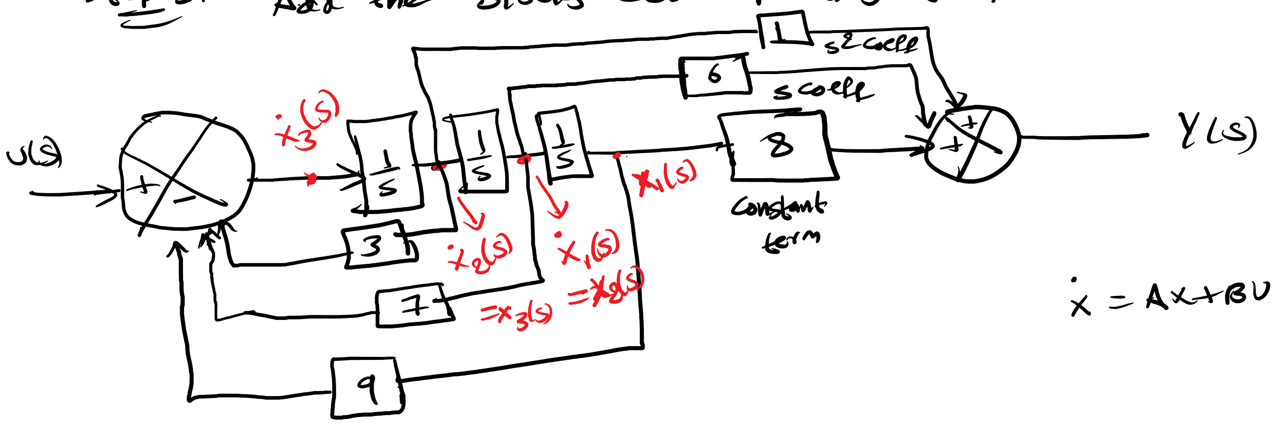
Problem:- Convert the transfer function $\frac{Y(s)}{U(s)} = \frac{5s^2+6s+8}{s^3+3s^2+7s+9}$ by using direct Decomposition Method.

Sol:- step 1:- $\frac{1}{s^3+3s^2+7s+9}\,(5s^2+6s+8)$

step 2:- $\frac{1}{s^3+3s^2+7s+9} = \frac{1}{(1)s^3+3s^2+7s+9}$



step 3:- Add the Blocks Corresponding to Numerator (s^2+6s+8)



$\dot{x} = Ax + BU$

State Equation
$$\begin{cases} \dot{x}_3(s) = -3x_3(s) - 7x_2(s) - 9x_1(s) + U(s) \\ \dot{x}_2(s) = x_3(s) \\ \dot{x}_1(s) = x_2(s) \end{cases}$$

$Y = Ax + BU$

o/p Equation
$$\{ \; Y = 8x_1(s) + 6x_2(s) + 1 \cdot x_3(s)$$

State Equation
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & -7 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{B} U$$

o/p Equation
$$[Y] = \underbrace{[8 \; 6 \; 1]}_{C} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{0}_{D}$$

② Obtain state model for the given transfer function using direct decomposition approach. $\frac{Y(s)}{U(s)} = \frac{2s^2+s+5}{s^3+6s^2+11s+4}$

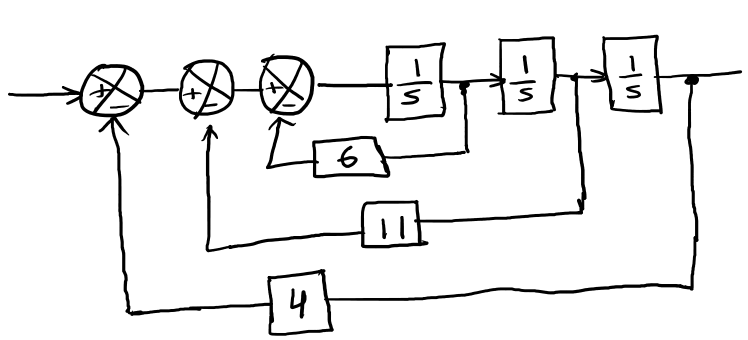
Sol:- step 1:- Rearrange the given transfer function.

$$\frac{Y(s)}{U(s)} = \frac{1}{s^3+6s^2+11s+4} \quad (2s^2+s+5)$$

step 2: Determine the order of transfer function.

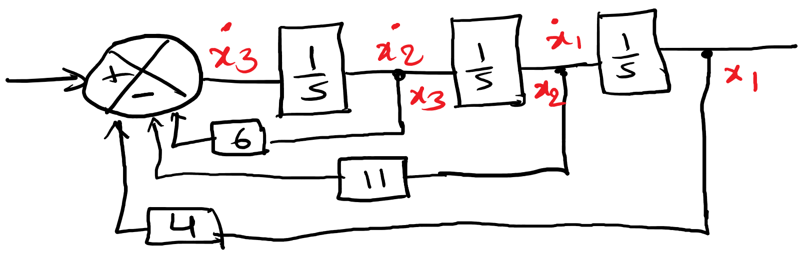
Here order is 3.

so use three $\frac{1}{s}$ block (Integrator) with 3 adders and feedback elements 6, 11, 4



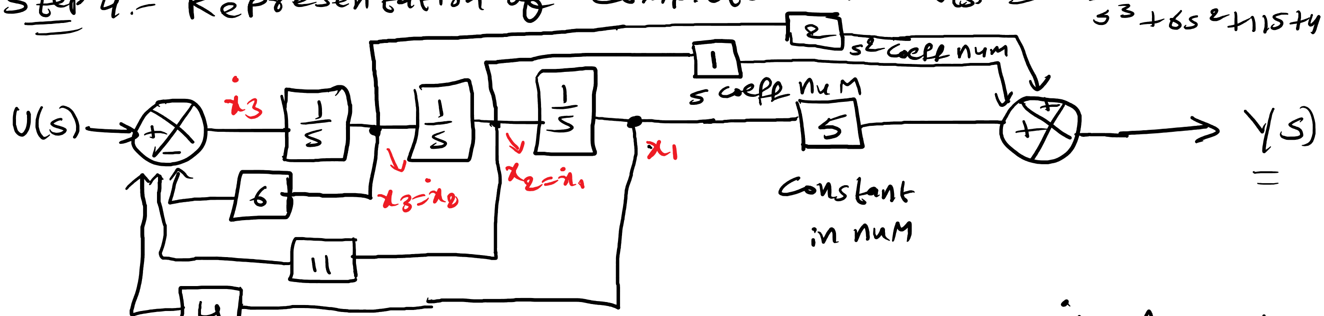
Denominator

step 3:- Combine adders and represent state variables.
order is 3 ; so represent 3 state variables



$$\begin{cases} x_2 = \dot{x}_1 \\ x_3 = \dot{x}_2 \end{cases}$$

step 4:- Representation of Complete T.F $\frac{Y(s)}{V(s)} = \frac{2s^2+s+5}{s^3+6s^2+11s+4}$



$\dot{X} = AX + BU$

$Y = CX + DU$

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = x_3 \\ \dot{x}_3 = -4x_1 - 11x_2 - 6x_3 + U \end{cases} \rightarrow \text{State Equation}$$

$y = 5x_1 + x_2 + 2x_3$ } → output equation

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \dots & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} U$$ } State

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} \cdot & \cdot & \cdot \\ 0 & 0 & 1 \\ -4 & -11 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \cdot \\ 0 \\ 1 \end{bmatrix} u$$ state Equation

$$\dot{x} = \underbrace{A}\ \tilde{x} + \tilde{B}\ \tilde{u}$$

state model

$$y = \begin{bmatrix} 5 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underset{\sim}{0}\, u$$ output Equation

$$Y = C\ X + D\ U$$

Method 2:- Fosters form or canonical form.

① obtain state model for T.F $\frac{Y(s)}{U(s)} = \frac{s^2+4}{(s+1)(s+2)(s+3)}$ using Foster form or canonical form.

sol:- step 1: Apply the partial fractions

$$\frac{Y(s)}{U(s)} = \frac{s^2+4}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}$$

$$= \frac{A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)}{(s+1)(s+2)(s+3)}$$

Equating numerators.

$$s^2+4 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$$

① Let $s=-1$

$$5 = A(1)(2) \Rightarrow A = \frac{5}{2} = 2.5$$

② Let $s=-2$

$$8 = B(-1)(1) \Rightarrow 8 = -B \Rightarrow B = -8.$$

③ Let $s=-3$

$$13 = C(-2)(-1) \Rightarrow 13 = 2C \Rightarrow C = \frac{13}{2} = 6.5$$

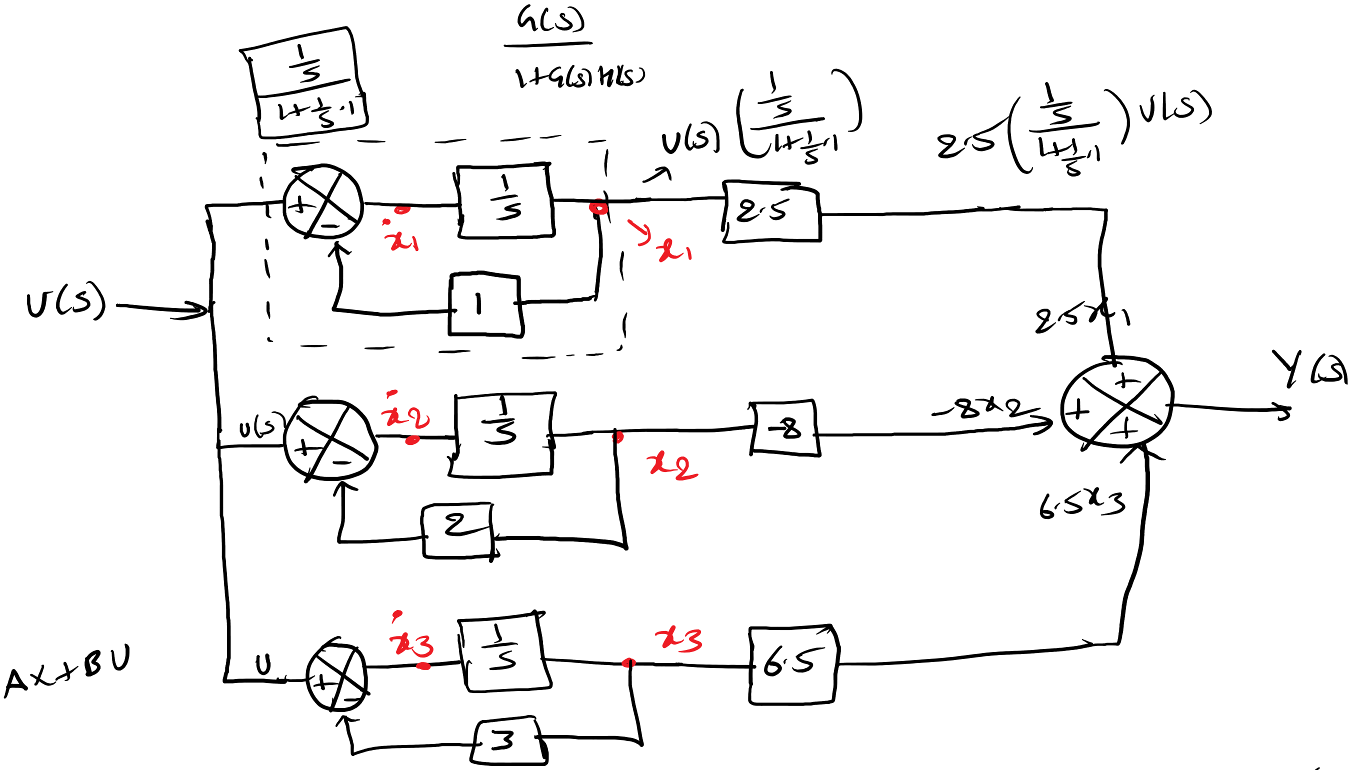
$$Y(s) \quad \_\_\_\_ \quad -\frac{8}{s+2} + \frac{6.5}{s+3}$$

Step 2 :- $Y(s) = \frac{2.5}{(s+1)} U(s) - \frac{8}{s+2} U(s) + \frac{6.5}{s+3} U(s)$

$= 2.5\left(\frac{\frac{1}{s}}{1+\frac{1}{s}\cdot 1}\right) U(s) + (-8)\left(\frac{\frac{1}{s}}{1+\frac{1}{s}\cdot 2}\right) U(s) + 6.5\left(\frac{\frac{1}{s}}{1+\frac{1}{s}\cdot 3}\right) U(s)$

$\downarrow$

$\frac{G(s)}{1+G(s)H(s)}$



$\dot{x} = Ax + BU$

$Y = 2.5x_1 - 8x_2 + 6.5x_3$

O/P equation

$\left.\begin{aligned} \dot{x}_1 &= -x_1 + U \\ \dot{x}_2 &= -2x_2 + U \\ \dot{x}_3 &= -3x_3 + U \end{aligned}\right\}$ State Equation

State Model $\quad \dot{x} = Ax + BU$ — State Equation

$Y = Cx + DU \rightarrow$ O/P Equation

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} U$

$\dot{x} = A \quad X \quad + B\,U$

$\frac{s^2+4}{(s+1)(s+2)(s+3)}$

$-1 \quad -2 \quad -3$

Poles

Diagonal elements

$y = [2.5 \;\; -8 \;\; 6.5] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (0)U$

$$Y = \begin{bmatrix} 2.5 & -8 & 6.5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + (0)U$$

$$C \quad X \quad + \quad D \quad U$$

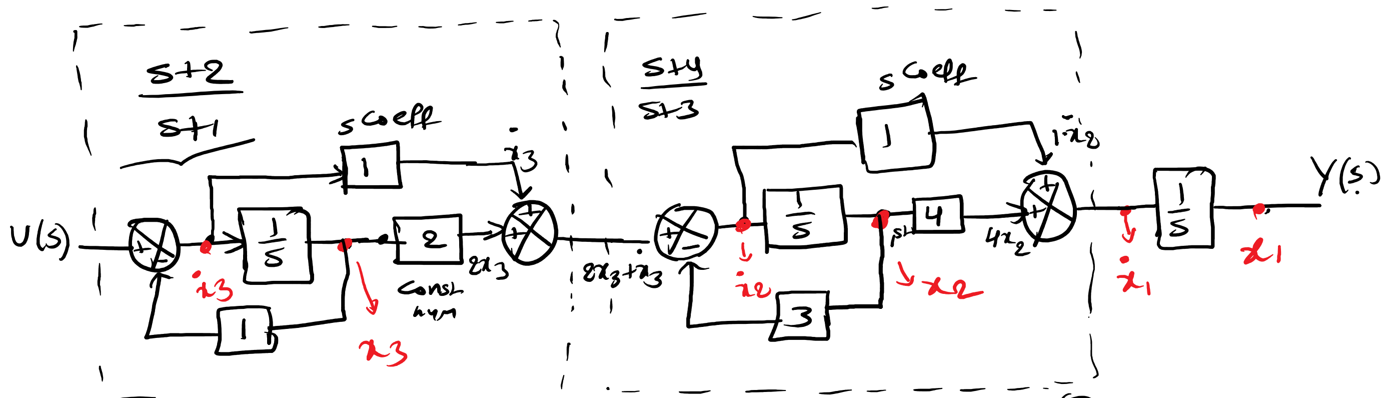
Method 3: cascade programming / Pole zero form

① Obtain state model for the TF $\frac{Y(s)}{U(s)} = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$ using cascade programming method.

sol:- **step 1:-** Rearrange the Transfer function as different groups

$$\frac{Y(s)}{V(s)} = \frac{(s+2)(s+4)}{s(s+1)(s+3)} = \underbrace{\frac{(s+2)}{(s+1)}}_{\text{Group 1}} \cdot \underbrace{\frac{(s+4)}{(s+3)}}_{\text{Group 2}} \cdot \underbrace{\frac{1}{s}}_{\text{Group3}}$$

Step 2:- Represent group 1, group 2 and group 3 in block diagram form using direct decomposition



$\dot{X} = AX+BU$
$Y = CX+DU$

From ② & ③ in ①

$$\dot{x}_1 = 4x_2 + \dot{x}_2 \quad —① \Rightarrow \quad 4x_2 - 3x_2 + 2x_3 - x_3 + U = \dot{x}_1$$

$$x_2 + x_3 + U = \dot{x}_1 \quad —④$$

$$\dot{x}_2 = -3x_2 + 2x_3 + \dot{x}_3 \quad —② \Rightarrow \quad \dot{x}_2 = -3x_2 + 2x_3 - x_3 + U$$

$$\dot{x}_2 = -3x_2 + x_3 + U \quad —⑤$$

$$\dot{x}_3 = -x_3 + U \quad —③$$

$$\left.\begin{aligned} \dot{x}_1 &= x_2 + x_3 + U \\ \dot{x}_2 &= -3x_2 + x_3 + U \end{aligned}\right\} \text{State Equation}$$

$$\left.\begin{aligned}\dot{x}_1 &= x_2 + x_3 + u \\ \dot{x}_2 &= -3x_2 + x_3 + u \\ \dot{x}_3 &= -x_3 + u\end{aligned}\right\}\text{ State Equation}$$

$$y = x_1 \quad \} \text{ output Equation}$$

State Model $\quad \dot{X} = AX + BU \qquad Y = CX + DU$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1 \\ x_2 \\ x_3\end{bmatrix} + \begin{bmatrix}1 \\ 1 \\ 1\end{bmatrix} u$$